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RETRACTIVE PRODUCT SPACES

Jose L. BLASCO

Universidad de Valencia, Departamento de Analisis Matematico, Dr. Moliner 50, Burjasot, Valencia, Spain

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A completely regular Hausdorff space X is called retractive if there is a retraction from βX onto $\beta X\backslash X$. A product space is retractive if and only if all factors are compact but one which is retractive.

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retraction	βX	product space
retractive	$\beta X\backslash X$	pseudocompact

Introduction

All spaces are taken to be completely regular Hausdorff spaces, βX denotes the Stone–Čech compactification of X, and X^* denotes the growth $\beta X\backslash X$. We will call a space *retractive* if there is a retraction of βX onto X^*. In [1], Comfort proves that the product of two retractive spaces need not be retractive and asks himself whether there exist two retractive spaces whose product also is retractive.

In this paper we prove that the product of two noncompact spaces is not retractive, herewith solving the question posed by Comfort. As a consequence of this result we characterize when the product of an arbitrary family of spaces is retractive.

The results

A space X is *pseudocompact* if each real-valued continuous function on X is bounded. We will use the following characterization of pseudocompact spaces [3, p. 370]: A space is pseudocompact if and only if every sequence of nonvoid open sets has a cluster point.

We shall need the following result about retractive spaces: (Comfort) A retractive space is pseudocompact. This result is proved from the continuum hypothesis (CH) in [5, Theorem 6.6]. See [2, Corollary] and [4, Theorem 0.1] for other proofs without using CH. Write N for the discrete space of all positive integers.

Theorem 1. *If X and Y are noncompact spaces, then the product $X \times Y$ is not retractive.*

Proof. Suppose that $X \times Y$ is retractive. Then $X \times Y$ is pseudocompact and from Glicksberg's theorem [3, Theorem 1], it follows that $\beta(X \times Y) = \beta X \times \beta Y$. Consequently the growth of $X \times Y$ is $(X^* \times Y) \cup (X^* \times Y^*) \cup (X \times Y^*)$. Now assume that r is a retraction of $\beta(X \times Y)$ onto $(X \times Y)^*$. We will define inductively the sequences (x_n), (y_n), (U_n), (U'_n), (V_n), (V'_n), (A_n) and (B_n) with the following properties for all $n \geq 1$:

(1) U_n and U'_n (respectively V_n and V'_n) are compact neighborhoods of x_n (respectively y_n) such that $U_n \subset U'_n \subset X$ (respectively $V_n \subset V'_n \subset Y$).

(2) A_n (respectively B_n) is a compact neighborhood of X^* (respectively Y^*) such that $A_{n+1} \subset A_n$ (respectively $B_{n+1} \subset B_n$).

(3) $A_n \cap U'_n = \emptyset$ and $U'_{n+1} \subset A_n$.

(4) $B_{n+1} \cap V'_n = \emptyset$ and $V'_n \subset B_n$.

(5) $r(A_n \times V_n) \subset X^* \times V'_n$ and $r(U_n \times B_n) \subset U'_n \times Y^*$.

(6) $r(x_{n+1}, y_n) \in X^* \times V_n$.

We describe the first step now. Let us note that $(X \times Y)^*$ is closed in $\beta(X \times Y)$, so $X \times Y$ is locally compact. Hence X and Y are locally compact spaces.

Choose $x_1 \in X$ and let U'_1 be a compact neighborhood of x_1, $U'_1 \subset X$. By continuity, $r^{-1}(U'_1 \times Y^*)$ is a neighborhood of $\{x_1\} \times Y^*$, therefore we can find compact neighborhoods B_1 and U_1 of Y^* and x_1, respectively, such that $U_1 \subset U'_1$ and $r(U_1 \times B_1) \subset U'_1 \times Y^*$.

Choose $y_1 \in Y \cap \operatorname{int}_{\beta Y} B_1$ and let V'_1 be a compact neighborhood of y_1, $V'_1 \subset Y \cap B_1$. By continuity, $r^{-1}(X^* \times V'_1)$ is a neighborhood of $X^* \times \{y_1\}$, therefore we can find compact neighborhoods A_1 and V_1 of X^* and y_1, respectively, such that

$$A_1 \cap U'_1 = \emptyset, \qquad V_1 \subset V'_1, \qquad r(A_1 \times V_1) \subset X^* \times V'_1.$$

Assume that the construction is made for $1, \ldots, n$. Then $r(A_n \times V_n) \subset X^* \times V'_n$ by (5). Choose $x_{n+1} \in X \cap \operatorname{int}_{\beta X} A_n$ such that $r(x_{n+1}, y_n) \in X^* \times V_n$. Let U'_{n+1} be a compact neighborhood of x_{n+1}, $U'_{n+1} \subset A_n \cap X$. As above, there exist compact neighborhoods B_{n+1} and U_{n+1} of Y^* and x_{n+1}, respectively, such that

$$U_{n+1} \subset U'_{n+1}, \qquad B_{n+1} \subset B_n, \qquad B_{n+1} \cap V'_n = \emptyset,$$

$$r(U_{n+1} \times B_{n+1}) \subset U'_{n+1} \times Y.$$

Choose $y_{n+1} \in Y \cap \operatorname{int}_{\beta Y} B_{n+1}$. Given a compact neighborhood V'_{n+1} of y_{n+1}, $V'_{n+1} \subset Y \cap B_{n+1}$, we can find compact neighborhoods A_{n+1} and V_{n+1} of X^* and y_{n+1}, respectively, such that

$$V_{n+1} \subset V'_{n+1}, \qquad A_{n+1} \subset A_n, \qquad A_{n+1} \cap U'_{n+1} = \emptyset,$$

$$r(A_{n+1} \times V_{n+1}) \subset X^* \times V'_{n+1}.$$

Now the inductive process is complete.

Claim. (a) *If* (u, v) *is a cluster point of the sequence* $((x_{n+1}, y_n))$, *then* $r(u, v) \in X^* \times Y^*$.

(b) *Let* $((u_j, v_j))$ *be a sequence such that* $u_j \in U_{n_j}$, $v_j \in V_{m_j}$ *and* $n_j \leqslant m_j < n_{j+1}$ *for every* $j \in N$. *If* (u, v) *is a cluster point of that sequence, then* $r(u, v) \in X^* \times Y^*$.

Proof. (a) From (5) it follows that the sequence $(r(x_{n+1}, y_n))$ is contained in $X^* \times Y$, therefore $r(u, v) \in \operatorname{cl}(X^* \times Y)$, where cl denotes the closure operator in $(X \times Y)^*$.

On the other hand, because $\beta(X \times Y) = \beta X \times \beta Y$ one can immediately check that (u, v) belongs to the closure in $\beta(X \times Y)$ of the set

$$\{(x_j, y_k) \colon k \geqslant j \geqslant 2\}.$$

From (2), (4) and (5), the set $\{r(x_j, y_k) \colon k \geqslant j \geqslant 2\}$ is contained in $X \times Y^*$, therefore $r(u, v) \in \operatorname{cl}(X \times Y^*)$. Since

$$\operatorname{cl}(X \times Y^*) \cap \operatorname{cl}(X^* \times Y) = X^* \times Y^*$$

we have that $r(u, v) \in X^* \times Y^*$.

Part (b) can be proved with a similar argument since (by (5)) $r(u_j, v_j) \in X \times Y^*$ for every $j \in N$ and $r(u_k, v_j) \in X^* \times Y$ for every $k > j \geqslant 2$.

The claim is proved.

The space X is pseudocompact since it is a continuous image of $X \times Y$. Hence, the family $(\operatorname{int}_X U_n)$ is not locally finite in X. Moreover, X is locally compact, therefore there exists a compact set $K \subset X$ such that the set $P_n = \operatorname{int}_X (K \cap U_n)$ is nonempty, for infinitely many integers n.

On the other hand, from (6) and part (a) of the claim it follows that each neighborhood of Y^* in βY intersects infinitely many sets V_n.

Choose a positive integer n_1 such that $P_{n_1} \neq \emptyset$. Given $t_1 \in P_{n_1}$, we find a compact neighborhood Q_1 of Y^* such that

$$r(\{t_1\} \times Q_1) \subset P_{n_1} \times Y^*.$$

Let m_1 be a positive integer such that $m_1 \geqslant n_1$ and the set $V_{m_1} \cap Q_1$ is nonempty. Now take $z_1 \in V_{m_1} \cap Q_1$.

Continuing by induction, we obtain the sequences $(t_j,)$, (z_j), (P_{n_j}) and (Q_j) such that for $j \geqslant 1$,

$$r(\{t_j\} \times Q_j) \subset P_{n_j} \times Y^* \subset K \times Y^*,$$

$$z_j \in V_{m_j} \cap Q_j, \qquad n_j \leqslant m_j < n_{j+1}.$$

If (u, v) is a cluster point of the sequence $((t_j, z_j))$, from part (b) of the claim it follows that $r(u, v) \in X^* \times Y^*$. However, this is impossible since the sequence $(r(t_j, z_j))$ is contained in the compact subset $K \times Y^*$ of $X \times Y^*$.

Thus, the assumption that r is a retraction of $\beta(X \times Y)$ onto $(X \times Y)^*$ leads to a contradiction. □

Corollary 2. *The product space $\prod\{X_i : i \in I\}$ is retractive if and only if all spaces X_i are compact but one which is retractive.*

Proof. The sufficiency follows from the fact that the product of a retractive space and a compact space is retractive [1, Theorem 3.4]. Let us prove the necessity of the condition stated.

In order to prove the result for I finite, we shall proceed by induction on the number of spaces. Suppose that $X_1 \times X_2$ is retractive. Then, from Theorem 1, one of these spaces, say X_1, is compact. Let us see now that X_2 is retractive. Since $X_1 \times X_2$ is pseudocompact, we have $\beta(X_1 \times X_2) = X_1 \times \beta X_2$ and $(X_1 \times X_2)^* = X_1 \times X_2^*$. Pick a point $x \in X_1$ and consider the map $h : (X_1 \times X_2)^* \to \{x\} \times X_2^*$ defined $h(u, v) = (x, v)$. If r is a retraction from $\beta(X_1 \times X_2)$ onto $(X_1 \times X_2)^*$, it is clear that $h \circ r$ is a retraction from $\{x\} \times \beta X_2$ onto $\{x\} \times X_2^*$. Hence X_2 is retractive.

Suppose now that the result holds for $n = 2, \ldots, m$. If $\prod\{X_i : 1 \leq i \leq m+1\}$ is retractive, either $\prod\{X_i : 1 \leq i \leq m\}$ is compact and X_{m+1} is retractive or $\prod\{X_i : 1 \leq i \leq m\}$ is retractive and X_{m+1} is compact. In the second case we can apply the induction hypothesis, and therefore the result holds for $m+1$ spaces.

Suppose now that I is infinite. From the hypothesis, the product space $\prod\{X_i : i \in I\}$ is locally compact, hence there exists a finite subset J of I such that X_i is compact for all $i \in I \setminus J$. It suffices then to apply the result (in the finite case) to the spaces $\{X_i : i \in J\}$ and $\prod\{X_i : i \in I \setminus J\}$. □

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